Dérivées partielles et changement de variables

exercice 1

Soit $\begin{array}{|llll|}\hline f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \\ \hline \end{array}$ *une fonction de classe* C^1. *On pose* $\begin{array}{|llll|}\hline g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & f(u+v,u-v) \\ \hline \end{array}$

Déterminer $\frac{\partial g}{\partial u}$ *et* $\frac{\partial g}{\partial v}$ *en fonction des dérivées partielles de* f.

exercice 2

Soit $\begin{array}{|llll|}\hline f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & f(x,y,z) \\ \hline \end{array}$ *une fonction de classe* C^1. *On pose* $\begin{array}{|llll|}\hline g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & f(u^2,v^2,u+2v) \\ \hline \end{array}$

Déterminer $\frac{\partial g}{\partial u}$ *et* $\frac{\partial g}{\partial v}$ *en fonction des dérivées partielles de* f.

exercice 3

Soit $\begin{array}{|llll|}\hline g: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & g(u,v) \\ \hline \end{array}$ *une fonction de classe* C^1. *On pose* $\begin{array}{|llll|}\hline f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g(x+y,xy) \\ \hline \end{array}$

Déterminer $\frac{\partial f}{\partial x}$ *et* $\frac{\partial f}{\partial y}$ *en fonction des dérivées partielles de* g.

exercice 4

Soit $\begin{array}{|llll|}\hline f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \\ \hline \end{array}$ *une fonction de classe* C^2. *On pose* $\begin{array}{|llll|}\hline g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & f(u+v,u-v) \\ \hline \end{array}$

Déterminer $\frac{\partial^2 g}{\partial u^2}, \frac{\partial^2 g}{\partial u \partial v}$ *et* $\frac{\partial^2 g}{\partial v^2}$ *en fonction des dérivées partielles de* f.

exercice 5

Soit $\begin{array}{|llll|}\hline g: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (u,v) & \longmapsto & g(u,v) \\ \hline \end{array}$ *une fonction de classe* C^2. *On pose* $\begin{array}{|llll|}\hline f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g(x+y,xy) \\ \hline \end{array}$

Déterminer $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}$ *et* $\dfrac{\partial^2 f}{\partial y^2}$ *en fonction des dérivées partielles de* g.

exercice 6

Soit $\begin{array}{|llll|}\hline f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \\ \hline \end{array}$ *une fonction de classe* C^2. *On pose* $\begin{array}{|llll|}\hline g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (r,\theta) & \longmapsto & f(r\cos\theta,r\sin\theta) \\ \hline \end{array}$

Déterminer $\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta}, \frac{\partial^2 g}{\partial r^2}, \frac{\partial^2 g}{\partial r \partial \theta}$ *et* $\frac{\partial^2 g}{\partial \theta^2}$ *en fonction des dérivées partielles de* f.

Solutions

résolution 1

- Pour tout $(u,v) \in \mathbb{R}^2$ on a

$$g(u,v) = f(u+v,u-v) = f(x,y)$$

- En utilisant la formule de dérivation d'une composée, on a

$$\begin{aligned}
\frac{\partial g}{\partial u}(u,v) &= 1.\partial_1 f(u+v,u-v) + 1.\partial_2 f(u+v,u-v) \\
&= 1.\frac{\partial f}{\partial x}(u+v,u-v) + 1.\frac{\partial f}{\partial y}(u+v,u-v) \\
&= \frac{\partial f}{\partial x}(u+v,u-v) + \frac{\partial f}{\partial y}(u+v,u-v)
\end{aligned}$$

et

$$\begin{aligned}
\frac{\partial g}{\partial v}(u,v) &= 1.\partial_1 f(u+v,u-v) - 1.\partial_2 f(u+v,u-v) \\
&= 1.\frac{\partial f}{\partial x}(u+v,u-v) - 1.\frac{\partial f}{\partial y}(u+v,u-v) \\
&= \frac{\partial f}{\partial x}(u+v,u-v) - \frac{\partial f}{\partial y}(u+v,u-v)
\end{aligned}$$

- On pouvait également écrire plus formellement

$$\frac{\partial g}{\partial u} = \frac{\partial x}{\partial u}.\partial_1 f + \frac{\partial y}{\partial u}.\partial_2 f \qquad \text{et} \qquad \frac{\partial g}{\partial v} = \frac{\partial x}{\partial v}.\partial_1 f + \frac{\partial y}{\partial v}.\partial_2 f$$

ce qui s'écrit encore

$$\frac{\partial g}{\partial u} = \frac{\partial x}{\partial u}.\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}.\frac{\partial f}{\partial y} \qquad \text{et} \qquad \frac{\partial g}{\partial v} = \frac{\partial x}{\partial v}.\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}.\frac{\partial f}{\partial y}$$

C'est à dire plus précisément

$$\begin{aligned}
\frac{\partial g}{\partial u}(u,v) &= \frac{\partial x}{\partial u}(u,v).\frac{\partial f}{\partial x}(x,y) + \frac{\partial y}{\partial u}(u,v).\frac{\partial f}{\partial y}(x,y) \\
&= 1.\frac{\partial f}{\partial x}(u+v,u-v) + 1.\frac{\partial f}{\partial y}(u+v,u-v) \\
&= \frac{\partial f}{\partial x}(u+v,u-v) + \frac{\partial f}{\partial y}(u+v,u-v)
\end{aligned}$$

et

$$\begin{aligned}
\frac{\partial g}{\partial v}(u,v) &= \frac{\partial x}{\partial v}(u,v).\frac{\partial f}{\partial x}(x,y) + \frac{\partial y}{\partial v}(u,v).\frac{\partial f}{\partial y}(x,y) \\
&= 1.\frac{\partial f}{\partial x}(u+v,u-v) - 1.\frac{\partial f}{\partial y}(u+v,u-v) \\
&= \frac{\partial f}{\partial x}(u+v,u-v) - \frac{\partial f}{\partial y}(u+v,u-v)
\end{aligned}$$

résolution 2

- Pour tout $(u,v) \in \mathbb{R}^2$ on a

$$g(u,v) = f(u^2,v^2,u+2v) = f(x,y,z)$$

- En utilisant la formule de dérivation d'une composée, on a

$$\begin{aligned}\frac{\partial g}{\partial u}(u,v) &= 2u.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + 0.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 1.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\\ &= 2u.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + \frac{\partial f}{\partial z}(u^2,v^2,u+2v)\end{aligned}$$

et

$$\begin{aligned}\frac{\partial g}{\partial v}(u,v) &= 0.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + 2v.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 2.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\\ &= 2v.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 2.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\end{aligned}$$

- On pouvait également écrire plus formellement

$$\frac{\partial g}{\partial u} = \frac{\partial x}{\partial u}.\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}.\frac{\partial f}{\partial y} + \frac{\partial z}{\partial u}.\frac{\partial f}{\partial z} \qquad \text{et} \qquad \frac{\partial g}{\partial v} = \frac{\partial x}{\partial v}.\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}.\frac{\partial f}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial f}{\partial z}$$

C'est à dire plus précisément

$$\begin{aligned}\frac{\partial g}{\partial u}(u,v) &= \frac{\partial x}{\partial u}(u,v).\frac{\partial f}{\partial x}(x,y) + \frac{\partial y}{\partial u}(u,v).\frac{\partial f}{\partial y}(x,y) + \frac{\partial z}{\partial u}(u,v).\frac{\partial f}{\partial z}(x,y)\\ &= 2u.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + 0.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 1.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\\ &= 2u.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + \frac{\partial f}{\partial z}(u^2,v^2,u+2v)\end{aligned}$$

et

$$\begin{aligned}\frac{\partial g}{\partial v}(u,v) &= \frac{\partial x}{\partial v}(u,v).\frac{\partial f}{\partial x}(x,y) + \frac{\partial y}{\partial v}(u,v).\frac{\partial f}{\partial y}(x,y) + \frac{\partial z}{\partial v}(u,v).\frac{\partial f}{\partial z}(x,y)\\ &= 0.\frac{\partial f}{\partial x}(u^2,v^2,u+2v) + 2v.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 2.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\\ &= 2v.\frac{\partial f}{\partial y}(u^2,v^2,u+2v) + 2.\frac{\partial f}{\partial z}(u^2,v^2,u+2v)\end{aligned}$$

résolution 3 • Pour tout $(x,y) \in \mathbb{R}^2$ on a

$$f(x,y) = g(x+y,xy) = g(u,v)$$

- En utilisant la formule de dérivation d'une composée, on a

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 1.\partial_1 g(x+y,xy) + y.\partial_2 g(x+y,xy)\\ &= 1.\frac{\partial g}{\partial u}(x+y,xy) + y.\frac{\partial g}{\partial v}(x+y,xy)\\ &= \frac{\partial g}{\partial u}(x+y,xy) + y.\frac{\partial g}{\partial v}(x+y,xy)\end{aligned}$$

et

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= 1.\partial_1 g(x+y,xy) + x.\partial_2 g(x+y,xy)\\ &= 1.\frac{\partial g}{\partial u}(x+y,xy) + x.\frac{\partial g}{\partial v}(x+y,xy)\\ &= \frac{\partial g}{\partial u}(x+y,xy) + x.\frac{\partial g}{\partial v}(x+y,xy)\end{aligned}$$

- On pouvait également écrire plus formellement

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}.\frac{\partial g}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial g}{\partial v} \qquad \text{et} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}.\frac{\partial g}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial g}{\partial v}$$

c'est à dire plus précisément

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\partial u}{\partial x}(x,y).\frac{\partial g}{\partial u}(u,v) + \frac{\partial v}{\partial x}(x,y).\frac{\partial g}{\partial v}(u,v)\\ &= 1.\frac{\partial g}{\partial u}(x+y,xy) + y.\frac{\partial g}{\partial v}(x+y,xy)\\ &= \frac{\partial g}{\partial u}(x+y,xy) + y.\frac{\partial g}{\partial v}(x+y,xy)\end{aligned}$$

et

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{\partial u}{\partial y}(x,y).\frac{\partial g}{\partial u}(u,v) + \frac{\partial v}{\partial y}(x,y).\frac{\partial g}{\partial v}(u,v)\\ &= 1.\frac{\partial g}{\partial u}(x+y,xy) + x.\frac{\partial g}{\partial v}(x+y,xy)\\ &= \frac{\partial g}{\partial u}(x+y,xy) + x.\frac{\partial g}{\partial v}(x+y,xy)\end{aligned}$$

résolution 4 • On a vu dans le calcul 1 que l'on a

$$\frac{\partial g}{\partial u}(u,v) = 1.\partial_1 f(u+v,u-v)+1.\partial_2 f(u+v,u-v) \qquad \text{et} \qquad \frac{\partial g}{\partial v}(u,v) = 1.\partial_1 f(u+v,u-v)-1.\partial_2 f(u+v,u-v)$$

- En utilisant la formule de dérivation d'une fonction composée, cela donne

$$\begin{aligned}\frac{\partial^2 g}{\partial u^2}(u,v) &= [1.\partial_1\partial_1 f(u+v,u-v)+1.\partial_2\partial_1 f(u+v,u-v)]+[1.\partial_1\partial_2 f(u+v,u-v)+1.\partial_2\partial_2 f(u+v,u-v)\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)+\frac{\partial^2 f}{\partial y\partial x}(u+v,u-v)+\frac{\partial^2 f}{\partial x\partial y}(u+v,u-v)+\frac{\partial^f}{\partial y^2}(u+v,u-v)\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)+2\frac{\partial^2 f}{\partial x\partial y}(u+v,u-v)+\frac{\partial^f}{\partial y^2}(u+v,u-v)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 g}{\partial u\partial v}(u,v) &= [1.\partial_1\partial_1 f(u+v,u-v)+1.\partial_2\partial_1 f(u+v,u-v)]-[1.\partial_1\partial_2 f(u+v,u-v)+1.\partial_2\partial_2 f(u+v,u-\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)+\frac{\partial^2 f}{\partial y\partial x}(u+v,u-v)-\frac{\partial^2 f}{\partial x\partial y}(u+v,u-v)-\frac{\partial^f}{\partial y^2}(u+v,u-v)\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)-\frac{\partial^f}{\partial y^2}(u+v,u-v)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 g}{\partial v^2}(u,v) &= [1.\partial_1\partial_1 f(u+v,u-v)-1.\partial_2\partial_1 f(u+v,u-v)]-[1.\partial_1\partial_2 f(u+v,u-v)-1.\partial_2\partial_2 f(u+v,u-v)\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)-\frac{\partial^2 f}{\partial y\partial x}(u+v,u-v)-\frac{\partial^2 f}{\partial x\partial y}(u+v,u-v)+\frac{\partial^f}{\partial y^2}(u+v,u-v)\\ &= \frac{\partial^2 f}{\partial x^2}(u+v,u-v)-2\frac{\partial^2 f}{\partial x\partial y}(u+v,u-v)+\frac{\partial^f}{\partial y^2}(u+v,u-v)\end{aligned}$$

remarque: comme f est de classe C^2, d'après le théorème de Schwarz,
on peut affirmer que $\dfrac{\partial^2 f}{\partial x\partial y} = \dfrac{\partial^2 f}{\partial y\partial x}$

résolution 5 • On a vu dans le calcul 3 que l'on a

$$\frac{\partial f}{\partial x}(x,y) = 1.\partial_1 g(x+y,xy)+y.\partial_2 g(x+y,xy) \qquad \text{et} \qquad \frac{\partial f}{\partial y}(x,y) = 1.\partial_1 g(x+y,xy)+x.\partial_2 g(x+y,xy)$$

• En utilisant la formule de dérivation d'une fonction composée, cela donne

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x,y) &= [\partial_1\partial_1 g(x+y,xy) + y.\partial_2\partial_1 g(x+y,xy)] + y.\,[\partial_1\partial_2 g(x+y,xy) + y.\partial_2\partial_2 g(x+y,xy)] \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + y\frac{\partial^2 g}{\partial v\partial u}(x+y,xy) + y\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + y^2\frac{\partial^2 g}{\partial v^2}(x+y,xy) \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + 2y\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + y^2\frac{\partial^2 g}{\partial v^2}(x+y,xy)\end{aligned}$$

$$\begin{aligned}\frac{\partial^f}{\partial y^2}(x,y) &= [\partial_1\partial_1 g(x+y,xy) + x.\partial_2\partial_1 g(x+y,xy)] + x.\,[\partial_1\partial_2 g(x+y,xy) + x.\partial_2\partial_2 g(x+y,xy)] \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + x\frac{\partial^2 g}{\partial v\partial u}(x+y,xy) + x\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + x^2\frac{\partial^2 g}{\partial v^2}(x+y,xy) \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + 2x\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + x^2\frac{\partial^2 g}{\partial v^2}(x+y,xy)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y}(x,y) &= [\partial_1\partial_1 g(x+y,xy) + y.\partial_2\partial_1 g(x+y,xy)] + x.\,[\partial_1\partial_2 g(x+y,xy) + y.\partial_2\partial_2 g(x+y,xy)] \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + y\frac{\partial^2 g}{\partial v\partial u}(x+y,xy) + x\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + xy\frac{\partial^2 g}{\partial v^2}(x+y,xy) \\ &= \frac{\partial^2 g}{\partial u^2}(x+y,xy) + (x+y)\frac{\partial^2 g}{\partial u\partial v}(x+y,xy) + xy\frac{\partial^2 g}{\partial v^2}(x+y,xy)\end{aligned}$$

remarque: comme g est de classe C^2, d'après le théorème de Schwarz, on peut affirmer que $\partial_1,\partial_2 g = \partial_2\partial_1 g$

résolution 6

- On a

$$\frac{\partial g}{\partial r}(r,\theta) = \cos\theta.\partial_1 f(r\cos\theta,r\sin\theta) + \sin\theta.\partial_2 f(r\cos\theta,r\sin\theta)$$

et

$$\frac{\partial g}{\partial \theta}(r,\theta) = -r\sin\theta.\partial_1 f(r\cos\theta,r\sin\theta) + r\cos\theta.\partial_2 f(r\cos\theta,r\sin\theta)$$

- et subséquemment

$$\begin{aligned}\frac{\partial^2 g}{\partial r^2}(r,\theta) &= \cos\theta.\left[\cos\theta.\partial_1 f\partial_1 f(r\cos\theta,r\sin\theta) + \sin\theta.\partial_2 f\partial_1 f(r\cos\theta,r\sin\theta)\right] \\ &+ \sin\theta.\left[\cos\theta.\partial_1 f\partial_2 f(r\cos\theta,r\sin\theta) + \sin\theta.\partial_2 f\partial_2 f(r\cos\theta,r\sin\theta)\right] \\ &= \cos^2\theta\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta) + 2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta,r\sin\theta) + \sin^2\theta\frac{\partial^f}{\partial y^2}(r\cos\theta,r\sin\theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 g}{\partial r\partial\theta}(r,\theta) &= -\sin\theta.\partial_1 f(r\cos\theta,r\sin\theta) - r\sin\theta.\left[\cos\theta.\partial_1\partial_1 f(r\cos\theta,r\sin\theta) + \sin\theta.\partial_2\partial_1 f(r\cos\theta,r\sin\theta)\right] \\ &+ \cos\theta.\partial_2 f(r\cos\theta,r\sin\theta) + r\cos\theta.\left[\cos\theta.\partial_1\partial_2 f(r\cos\theta,r\sin\theta) + \sin\theta.\partial_2\partial_2 f(r\cos\theta,r\sin\theta)\right] \\ &= -\sin\theta\frac{\partial f}{\partial x}(..) + \cos\theta\frac{\partial f}{\partial y}(..) + r\cos\theta\sin\theta\left(\frac{\partial^f}{\partial y^2}(..) - \frac{\partial^2 f}{\partial x^2}(..)\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 f}{\partial x\partial y}(..)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 g}{\partial \theta^2}(r,\theta) &= -r\cos\theta.\partial_1 f(r\cos\theta,r\sin\theta) - r\sin\theta\left[-r\sin\theta.\partial_1\partial_1 f(r\cos\theta,r\sin\theta) + r\cos\theta.\partial_2\partial_1 f(r\cos\theta,r\sin\theta\right. \\ &- r\sin\theta\partial_2 f + r\cos\theta\left[-r\sin\theta.\partial_1\partial_2 f(r\cos\theta,r\sin\theta) + r\cos\theta.\partial_2\partial_2 f(r\cos\theta,r\sin\theta)\right] \\ &= -r\cos\theta\frac{\partial f}{\partial x}(..) - r\sin\theta\frac{\partial f}{\partial y}(..) + r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2}(..) + r^2\cos^2\theta\frac{\partial^f}{\partial y^2}(..) - 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(..)\end{aligned}$$

- *remarque: comme f est de classe C^2, d'après le théorème de Schwarz, on peut affirmer que* $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$
- *vous pouvez maintenant aisément retrouver la formule du laplacien en polaire*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^f}{\partial y^2} = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}.\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}$$